

# Contents

<b>Preface</b>	vii
<b>Brief Contents</b>	ix
<b>Cross-Reference Codes</b>	xi
<b>Introduction</b>	1
<b>Glossary</b>	7
<b>Part 1: Classical Fields and Supersymmetry</b>	
<b>Pierre Deligne and John W. Morgan, Notes on Supersymmetry (following Joseph Bernstein)</b>	41
<b>Chapter 1. Multilinear Algebra</b>	45
§1.1. The sign rule	45
§1.2. Categorical approach	47
§1.3. Examples of the categorical approach	48
§1.4. Free modules	53
§1.5. Free commutative algebras	54
§1.6. The trace	54
§1.7. Even rules	56
§1.8. Examples of the “even rules” principle	58
§1.9. Alternate description of super Lie algebras	59
§1.10. The Berezinian of an automorphism	59
§1.11. The Berezinian of a free module	61
<b>Appendix. Graded super vector spaces</b>	62

Chapter 2. Super Manifolds: Definitions	65
§2.1–§2.7. Super manifolds as ringed spaces	65
§2.8–§2.9. The functor of points approach to super manifolds	68
§2.10. Super Lie groups	69
§2.11. Classical series of super Lie groups	70
Chapter 3. Differential Geometry of Super Manifolds	71
§3.1. Introduction	71
§3.2. Vector bundles	71
§3.3. The tangent bundle, the cotangent bundle and the de Rham complex	72
§3.4. The inverse and implicit function theorems	75
§3.5. Distributions	75
§3.6. Connections on vector bundles	77
§3.7. Actions of super Lie algebras; vector fields and flows; Lie derivative	78
§3.8. Super Lie groups and Harish-Chandra pairs	79
§3.9. Densities	80
§3.10. Change of variables formula for densities	80
§3.11. The Lie derivative of sections of $\text{Ber}(\Omega_M^1)$	83
§3.12. Integral forms	84
§3.13. A second definition of integral forms	85
§3.14. Generalized functions	86
§3.15. Integral forms as functions of infinitesimal submanifold elements	86
Chapter 4. Real Structures	89
§4.1–§4.3. Real structures and $*$ -operations	89
§4.4. Super Hilbert spaces	90
§4.5. SUSY quantum mechanics	91
§4.6. Real and complex super manifolds	92
§4.7. Complexification, in infinite dimensions	93
§4.8. cs manifolds	94
§4.9. Integration on cs manifolds; examples	94
References	97
<b>Pierre Deligne, Notes on Spinors</b>	99
Chapter 1. Overview	101
Chapter 2. Clifford Modules	107
Chapter 3. Reality of Spinorial Representations and Signature Modulo 8	113
Chapter 4. Pairings and Dimension Modulo 8, Over $\mathbb{C}$	119

Chapter 5. Passage to Quadratic Spaces	127
Chapter 6. The Minkowski Case	129
References	135
<b>Pierre Deligne and Daniel S. Freed, Classical Field Theory</b>	<b>137</b>
Chapter 1. Classical Mechanics	143
§1.1. The nonrelativistic particle	143
§1.2. The relativistic particle	146
§1.3. Noether's theorem	147
§1.4. Synthesis	150
Chapter 2. Lagrangian Theory of Classical Fields	153
§2.1. Dimensional analysis	153
§2.2. Densities and twisted differential forms	154
§2.3. Fields and lagrangians	155
§2.4. First order lagrangians	162
§2.5. Hamiltonian theory	163
§2.6. Symmetries and Noether's theorem	165
§2.7. More on symmetries	171
§2.8. Computing Noether's current by gauging symmetries	175
§2.9. The energy-momentum tensor	178
§2.10. Finite energy configurations, classical vacua, and solitons	183
§2.11. Dimensional reduction	187
Appendix: Takens' acyclicity theorem	188
Chapter 3. Free Field Theories	191
§3.1. Coordinates on Minkowski spacetime	191
§3.2. Real scalar fields	192
§3.3. Complex scalar fields	194
§3.4. Spinor fields	195
§3.5. Abelian gauge fields	198
Chapter 4. Gauge Theory	201
§4.1. Classical electromagnetism	201
§4.2. Principal bundles and connections	204
§4.3. Pure Yang-Mills Theory	207
§4.4. Electric and magnetic charge	209
Chapter 5. $\sigma$ -Models and Coupled Gauge Theories	211
§5.1. Nonlinear $\sigma$ -models	211
§5.2. Gauge theory with bosonic matter	213

Chapter 6. Topological Terms	215
§6.1. Gauge theory	215
§6.2. Wess-Zumino-Witten terms	217
§6.3. Smooth Deligne cohomology	218
Chapter 7. Wick Rotation: From Minkowski Space to Euclidean Space	221
§7.1. Kinetic terms for bosons	222
§7.2. Potential terms	222
§7.3. Topological terms and $\theta$ -terms	223
§7.4. Kinetic terms for fermions	223
References	225
<b>Pierre Deligne, Daniel S. Freed, Supersolutions</b>	<b>227</b>
Chapter 1. Preliminary Topics	231
§1.1. Super Minkowski spaces and super Poincaré groups	231
§1.2. Superfields, component fields, and lagrangians	236
§1.3. A simple example	241
Chapter 2. Coordinates on Superspace	243
§2.1. $M^{3 2}$ , $M^{4 4}$ , $M^{6 (8,0)}$ and their complexifications	243
§2.2. Dimensional reduction	245
§2.3. Coordinates on $M^{3 2}$	246
§2.4. Coordinates on $M^{4 4}$	249
§2.5. Coordinates on $M^{6 (8,0)}$	254
§2.6. Low dimensions	257
Chapter 3. Supersymmetric $\sigma$ -Models	261
§3.1. Preliminary remarks on linear algebra	261
§3.2. The free supersymmetric $\sigma$ -model	263
§3.3. Nonlinear supersymmetric $\sigma$ -model	267
§3.4. Supersymmetric potential terms	273
§3.5. Superspace construction	276
§3.6. Dimensional reduction	276
Chapter 4. The Supersymmetric $\sigma$ -model in dimension 3	279
§4.1. Fields and supersymmetry transformations on $M^{3 2}$	279
§4.2. The $\sigma$ -model action on $M^{3 2}$	282
§4.3. The potential term on $M^{3 2}$	282
§4.4. Analysis of the classical theory	284
§4.5. Reduction to $M^{2 (1,1)}$	289
Chapter 5. The Supersymmetric $\sigma$ -Model in dimension 4	291
§5.1. Fields and supersymmetry transformations on $M^{4 4}$	291
§5.2. The $\sigma$ -model action on $M^{4 4}$	293
§5.3. The superpotential term on $M^{4 4}$	296
§5.4. Analysis of the classical theory	297

Chapter 6. Supersymmetric Yang-Mills Theories	299
§6.1. The minimal theory in components	299
§6.2. Gauge theories with matter	303
§6.3. Superspace construction	309
Chapter 7. $N = 1$ Yang-Mills Theory in Dimension 3	313
§7.1. Constrained connections on $M^{3 2}$	313
§7.2. The Yang-Mills action on $M^{3 2}$	317
§7.3. Gauge theory with matter on $M^{3 2}$	318
Chapter 8. $N = 1$ Yang-Mills Theory in Dimension 4	321
§8.1. Constrained connections on $M^{4 4}$	321
§8.2. The Yang-Mills action on $M^{4 4}$	325
§8.3. Gauge theory with matter on $M^{4 4}$	327
Chapter 9. $N = 2$ Yang-Mills in Dimension 2	331
§9.1. Dimensional reduction of bosonic Yang-Mills	331
§9.2. Constrained connections on $M^{2 (2,2)}$	333
§9.3. The reduced Yang-Mills action	334
Chapter 10. $N = 1$ Yang-Mills in Dimension 6 and $N = 2$ Yang-Mills in Dimension 4	337
§10.1. Constrained connections on $M^{6 (8,0)}$	337
§10.2. Reduction to $M^{4 8}$	340
§10.3. More theories on $M^{4 4}$ with extended supersymmetry	344
Chapter 11. The Vector Multiplet on $M^{6 (8,0)}$	347
§11.1. Complements on $M^{6 (8,0)}$	347
§11.2. Constrained connections	348
§11.3. An auxiliary Lie algebra	349
§11.4. Components of constrained connections	352
References	355
<b>Pierre Deligne and Daniel S. Freed, Sign Manifesto</b>	<b>357</b>
§1. Standard mathematical conventions	357
§2. Choices	357
§3. Rationale	358
§4. Notation	358
§5. Consequences of §2 on other signs	360
§6. Differential forms	362
§7. Miscellaneous signs	363

## Part 2: Formal Aspects of QFT

<b>Pierre Deligne, Note on Quantization</b>	<b>367</b>
<b>David Kazhdan, Introduction to QFT</b>	<b>377</b>
<b>(Notes by Roman Bezrukavnikov)</b>	
<b>Lecture 1. Wightman Axioms</b>	<b>379</b>
§1.0. Setup and notations	379
§1.1. Wightman axioms	380
§1.2. Wightman functions	380
§1.3. Reconstruction of QFT from Wightman functions	381
§1.4. Spin-statistics theorem	382
§1.5. Mass spectrum of a theory	383
§1.6. Asymptotics of Wightman functions	384
<b>Lecture 2. Euclidean Formulation of Wightman QFT</b>	<b>387</b>
§2.1. Analytic continuation of Wightman functions	387
§2.2. Euclidean formulation of Wightman QFT	390
§2.3. Schwinger functions and measures on the map-spaces	391
§2.4. PCT theorem	392
§2.5. Time-ordering	394
<b>Lecture 3. Free Field Theories</b>	<b>395</b>
§3.1. Some examples of free classical field theories	395
§3.2. Clifford module	396
§3.3. Examples of free QFT's	397
§3.4. Free QFT of arbitrary spin	398
§3.5. Wightman functions of a free field theory; truncated Wightman functions	401
§3.6. Gaussian measures	402
§3.7. Normal ordering	402
<b>Lecture 4. Scattering Theory</b>	<b>405</b>
§4.1. Introduction	405
§4.2. System of $n$ particles (potential scattering)	406
§4.3. Haag-Ruelle theory	407
§4.4. Scattering matrix	412
<b>Lecture 5. Feynman Graphs</b>	<b>413</b>
§5.1. Feynman graph expansion	413
§5.2. Quasi-classical (low-loop) approximations	416
§5.3. Effective potential	418

(Notes by Radu Constantinescu, Pavel Etingof,  
and David Kazhdan)

Lecture 1. Renormalization of Feynman Diagrams	421
§1.1. Perturbative expansion of a two-point correlation function	421
§1.2. The $\phi^3$ -theory	423
§1.3. Perturbative expansion of Feynman integrals	424
§1.4. Computation of a Feynman integral over functions on a Minkowski space	425
§1.5. Renormalization of divergent graphs	429
§1.6. Renormalization in higher orders	431
Lecture 2. Perturbative Renormalizability of Field Theories	435
§2.1. Renormalizability of quantum field theories	435
§2.2. Critical dimensions of some field theories	437
§2.3. Perturbative renormalization of critical theories	440
Lecture 3. Composite Operators and Operator Product Expansion	445
§3.1. Local functionals in a classical field theory	445
§3.2. Quantization of local functionals in a free theory	446
§3.3. Multiplication of composite operators	448
§3.4. Operator product expansion (OPE) in the free theory	449
§3.5. Normal ordering and renormalization	452
§3.6. Composite operators in an interacting critical theory	452
§3.7. Stability of the classical field equations under quantization	454
§3.8. Operator product expansion in an interacting theory	456
Lecture 4. Scattering Theory	461
§4.1. Nonrelativistic scattering theory: the asymptotic conditions	461
§4.2. Relation with experiments	462
§4.3. The Lippmann-Schwinger equation	463
§4.4. The Born approximation	464
§4.5. Feynman diagrams	465
§4.6. Relativistic versus non-relativistic scattering theory: propagation of particles	466
§4.7. Relativistic versus non-relativistic scattering theory: propagation of signals	467
Lecture 5. Remarks on Renormalization and Asymptotic Freedom	469
§5.1. Ambiguity in operator products	469
§5.2. Symmetry breaking	470
§5.3. An oversimplified version of experimental confirmation of asymptotic freedom	470

**(Notes by John Morgan)**

Lecture 1. The Dirac Operator in Finite Dimensions	477
§1.1. Introduction	477
§1.2. The Dirac operator on a spin manifold	478
§1.3. The case of a circle action	485
§1.4. $\sigma$ -models in $1 + 1$ dimensions	492
Lecture 2. The Dirac Operator on Loop Space	497
§2.1. Introduction	497
§2.2. The Lagrangian formulation: $\sigma$ -models in two dimensions	497
§2.3. Quantization	499
§2.4. The index of $Q_+$	500
§2.5. The computation around the fixed points of the $S^1$ -action	501
§2.6. Path integral approach	505
§2.7. Bundles whose coupled signature or Dirac operator has constant character	507
§2.8. Generalization to vector bundles over the loop space	509

**Ludwig Faddeev, Elementary Introduction to Quantum Field Theory** 513**(Notes by Lisa Jeffrey)**

Lecture 1. Basics of Quantum Mechanics and Canonical Quantization in Hilbert Space	515
§1.1. Observables and states	515
§1.2. Dynamics	517
§1.3. Quantization	519
Lecture 2. The Harmonic Oscillator and Free Fields	523
§2.1. The harmonic oscillator	523
§2.2. Perturbations	524
§2.3. Quantum field theory	526
§2.4. S-matrix and Feynman diagrams	529
Lecture 3. Comments on Scattering	531
§3.1. The S-matrix	531
§3.2. Mass renormalization	531
§3.3. Charge renormalization	534
Lecture 4. Singular Lagrangians	537
§4.1. Lagrangian and Hamiltonian formalisms	537
§4.2. Constraints	538
§4.3. Examples	539



Lecture 5. Quantization of Yang-Mills Fields	545
§5.1. The physical variables	545
§5.2. Gauge conditions in the functional integral	547
<b>David Gross, Renormalization Groups</b>	<b>551</b>
(Notes by Pavel Etingof and David Kazhdan)	
Lecture 1. Introduction to Renormalization Groups	553
§1.1. What is renormalization group?	553
§1.2. The general scheme of the method of renormalization group	554
§1.3. Wilsonian scheme for the theory of a scalar field: a mathematical description	555
§1.4. Applications of renormalization group theory to phase transitions	559
§1.5. Reminder of renormalization theory	559
§1.6. Dimensional regularization	560
Lecture 2. Renormalization Group Equation	563
§2.1. Finite renormalization	563
§2.2. The dimensional regularization prescription of finite renormalization	564
§2.3. Scale-dependence of finite renormalization prescriptions	565
§2.4. The renormalization group flow corresponding to a scale-dependent renormalization prescription	566
§2.5. Computation of the renormalization group flow in the 1-loop approximation	570
§2.6. Asymptotic freedom	570
Lecture 3. A Closer Look at the Renormalization Group Equation	573
§3.1. Dynamical patterns of the renormalization group flow	573
§3.2. Are there any asymptotically free theories without nonabelian gauge fields?	576
§3.3. Renormalization group equations with many couplings	578
§3.4. The renormalization group equation for composite operators	578
§3.5. Anomalous dimension	580
§3.6. The canonical part of the $\beta$ -function	583
Lecture 4. Dynamical Mass Generation and Symmetry Breaking in the Gross-Neveu Model	585
§4.1. Dynamical mass generation	585
§4.2. The Gross-Neveu model	587
§4.3. The large $N$ limit	588
Lecture 5. The Wilsonian Renormalization Group Equation	595

<b>Pavel Etingof, Note on Dimensional Regularization</b>	597
§1. The $D$ -dimensional integral	597
§2. $D$ -dimensional integral with parameters	601
§3. $D$ -dimensional integral of functions arising from Feynman diagrams	605
§4. Dimensional regularization of Feynman integrals	606
§5. $D$ -dimensional Stokes formula	607
<b>Edward Witten, Homework</b>	609
<b>(Solutions by Pierre Deligne, Daniel Freed, Lisa Jeffrey, and Siye Wu)</b>	
Chapter 1. Problems	611
Problem Sets from fall term	611
Fall exam	629
Superhomework	632
Addendum to superhomework	642
Chapter 2. Solutions to Selected Problems	647
<b>Index</b>	719